Math 121 2.4-1st Product Rule

Objective: i) Find the derivative of a function which is the product of two expressions which both contain x (or the variable)

The Product Rule

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

This is NOT the same as $f'(x)\cdot g'(x)$.

Where does this come from? The definition of the derivative, of course.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

except... That $f(x)$ is $f(x)\cdot g(x)$...

$$\lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

add and subtract a useful term...

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

separate...

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0}\frac{f(x)g(x+h) - f(x)g(x)}{h}$$

factor...

$$= \lim_{h\to 0} g(x+h)\left[\frac{f(x+h)-f(x)}{h}\right] + \lim_{h\to 0} f(x)\left[\frac{g(x+h)-g(x)}{h}\right]$$

separate...

$$= \left[\lim_{h\to 0} g(x+h)\right]\cdot\left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right] + \left[\lim_{h\to 0} f(x)\right]\cdot\left[\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right]$$

take limits...

$$= g(x)\cdot f'(x) + f(x)\cdot g'(x)$$

Find the derivative two ways.

① $\frac{d}{dx}(x^3 \cdot x^5)$

a) simplify first

$= \frac{d}{dx}(x^{3+5})$ exponent laws

$= \frac{d}{dx}(x^8)$

$= \boxed{8x^7}$ power rule

b) product rule

$\frac{d}{dx}(x^3 \cdot x^5)$

$= \frac{d}{dx}(x^3) \cdot x^5 + \frac{d}{dx}(x^5) \cdot x^3$ product rule

$= 3x^2 \cdot x^5 + 5x^4 \cdot x^3$ power rule

$= 3x^{2+5} + 5x^{4+3}$ exp law

$= 3x^7 + 5x^7$ combine.

$= \boxed{8x^7}$

Yes... in this case, it's much better to simplify first
In fact, it usually is better to simplify first.
Except... after 2.6 : you'll need to differentiate using the chain rule ... and then you have to use the product rule.

② $f(x) = x \cdot (x^2+1)^{1/2}$ ← order of operations prevents simplifying!

$f'(x) = \frac{d}{dx}(x) \cdot (x^2+1)^{1/2} + \underbrace{\frac{d}{dx}(x^2+1)^{1/2}} \cdot x$

This is a derivative using the Chain Rule which we have not done yet.

Find derivatives two ways
a) simplify first
b) product rule

③ $\frac{d}{dx}\left[(x^2-x+2)(x^3+3)\right]$

a) $\frac{d}{dx}\left[x^5+3x^2-x^4-3x+2x^3+6\right]$ multiply

$= \frac{d}{dx}\left[x^5-x^4+2x^3+3x^2-3x+6\right]$ standard form

$= \boxed{5x^4-4x^3+6x^2+6x-3}$ power rule + constant multiple rule

b) product rule

$= \frac{d}{dx}(x^2-x+2)\cdot(x^3+3) + \frac{d}{dx}(x^3+3)\cdot(x^2-x+2)$ product rule

$= (2x-1)(x^3+3) + (3x^2)(x^2-x+2)$ power rule + constant multiple

$= 2x^4+6x-x^3-3 + 3x^4-3x^3+6x^2$ FOIL + dist

$= \boxed{5x^4-4x^3+6x^2+6x-3}$ combine standard form

same as before.

④ $f(x)=\sqrt{x}\,(2x-4)$

a) $f(x)= x^{1/2}(2x-4)$

$= 2x^{1/2+1}-4x^{1/2}$

$= 2x^{3/2}-4x^{1/2}$

$f'(x)= 2\cdot\frac{3}{2}\cdot x^{3/2-1} - 4\cdot\frac{1}{2}\cdot x^{1/2-1}$

$= 3x^{1/2}-2x^{-1/2}$

$= \boxed{3\sqrt{x}-\frac{2}{\sqrt{x}}}$

b) $f(x) = x^{1/2}(2x-4)$

$f'(x) = \frac{d}{dx}(x^{1/2}) \cdot (2x-4) + \frac{d}{dx}(2x-4) \cdot x^{1/2}$ product rule

$= \frac{1}{2}x^{-1/2}(2x-4) + 2x^{1/2}$ power rule & constant multiple rule

$= \frac{1}{2}x^{-1/2} \cdot 2x - \frac{1}{2}x^{-1/2} \cdot 4 + 2x^{1/2}$ distribute

$= x^{-1/2+1} - 2x^{-1/2} + 2x^{1/2}$ exp laws

$= x^{1/2} - 2x^{-1/2} + 2x^{1/2}$ combine

$= 3x^{1/2} - 2x^{-1/2}$

$= \boxed{3\sqrt{x} - \frac{2}{\sqrt{x}}}$

Simplify and differentiate

5. $f(x) = \frac{3x^4}{8}$

$f(x) = \frac{3}{8}x^4$ coefficient in front

$f'(x) = \frac{3}{8} \cdot 4 \cdot x^3$ power rule and constant multiple rule

$\boxed{f'(x) = \frac{3}{2}x^3}$

6. $f(x) = \frac{3}{2x^2}$

$f(x) = \frac{3}{2}x^{-2}$ coefficient in front, negative exponent in numerator

$f'(x) = \frac{3}{2} \cdot (-2)x^{-2-1}$ power rule and constant multiple rule

$= -3x^{-3}$

$\boxed{f'(x) = \frac{-3}{x^3}}$

⑦ $f(x) = \dfrac{x^2 - 5x}{3x}$

$f(x) = \dfrac{x^2}{3x} - \dfrac{5x}{3x}$ divide each term

$f(x) = \frac{1}{3}x - \frac{5}{3}$ coefficients in front, reduce

$f'(x) = \frac{1}{3} \cdot 1 - 0$

$$\boxed{f'(x) = \frac{1}{3}}$$

⑧ $y = \dfrac{4}{\sqrt{x}}$

$y = 4x^{-1/2}$ write with negative exponent

$y' = 4 \cdot \left(-\frac{1}{2}\right) x^{-1/2 - 1}$

$y' = -2x^{-3/2}$

$$\boxed{y'(x) = \frac{-2}{x^{3/2}}}$$

$$\boxed{y'(x) = \frac{-2}{\sqrt{x^3}}}$$

Notation:

Note: When differentiating with respect to t instead of x, we write $\frac{d}{dt}$ instead.

⑨ Practice with product rule

$f(t) = 6t^{4/3}(3t^{2/3} + 1)$

$f'(t) = \frac{d}{dt}(6t^{4/3}) \cdot (3t^{2/3} + 1) + \frac{d}{dt}(3t^{2/3} + 1) \cdot 6t^{4/3}$

$= 6 \cdot \frac{4}{3} \cdot t^{4/3 - 1}(3t^{2/3} + 1) + \left(3 \cdot \frac{2}{3} t^{2/3 - 1} + 0\right) \cdot 6t^{4/3}$

$= 8t^{1/3}(3t^{2/3} + 1) + 2t^{-1/3} \cdot 6t^{4/3}$

$= 24t^{1/3+2/3} + 8t^{1/3} + 12t^{-1/3+4/3}$

$= 24t^{3/3} + 8t^{1/3} + 12t^{3/3}$

$= 24t + 8t^{1/3} + 12t$

$\boxed{f'(t) = 36t + 8t^{1/3}}$

10) Suppose that after x months, monthly sales of a music compact disk (CD) are predicted to be $S(x) = x^2(8-x^2)$ thousand, which is valid for $0 \leq x \leq 2$ months. Find the rate of change of sales after one month.

instantaneous rate of change when $x=1$
means derivative, evaluated when $x=1$.

find the derivative $S'(x)$ by either simplifying first or by using the product rule.

$S(x) = x^2(8-x^2)$

$S(x) = 8x^2 - x^4$ simplify

$S'(x) = 8 \cdot 2x^1 - 4x^3$

$S'(x) = 16x - 4x^3$

evaluate $S'(1) = 16(1) - 4(1)^3$

$= 16 - 4$

$= \boxed{12 \text{ CDs sold/mo. at the first month.}}$ thousand

$= \boxed{12000 \text{ CD sold/mo}}$

(11) From 2008 to 2011, new car loan interest rates at auto finance companies were approximately

$$I(x) = 0.45(x-1.7)(x^2-12.5x+43)$$

percent where x is the number of years after 2005.
Differentiate using the Product Rule.
Find $I'(5)$ and $I'(6)$
Interpret the results.

$$I'(x) = \underset{\substack{\uparrow \\ \text{constant multiple}}}{0.45}\left[\underbrace{\frac{d}{dx}(x-1.7)\cdot(x^2-12.5x+43) + \frac{d}{dx}(x^2-12.5x+43)\cdot(x-1.7)}_{\text{product rule}}\right]$$

$$= 0.45\left[1\cdot(x^2-12.5x+43) + (2x-12.5+0)\cdot(x-1.7)\right]$$

power rule & constant multiple rule

$$= 0.45\left[x^2-12.5x+43 + \underbrace{2x^2-3.4x-12.5x+21.25}_{\text{FOIL}}\right]$$

$$= 0.45(3x^2-28.4x+64.25) \quad \text{combine like terms}$$

$$\boxed{I'(x) = 1.35x^2-12.78x+28.9125}$$

$$I'(5) = 1.35(5)^2-12.78(5)+28.9125$$
$$= -1.2375\ \%\text{ per year}$$

$2005+5 = 2010.$

$\boxed{\text{Interest rates fell (negative) } 1.2375\ \%/\text{yr during } 2010}$

$$I'(6) = 1.35(6)^2-12.78(6)+28.9125$$
$$= .8325$$

$2005+6 = 2011$

$\boxed{\text{Interest rates rose (positive) } .8325\ \%/\text{yr during } 2011}$